ABOUT CANTOR'S UNCOUNTABILITY PROOFS

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In this paper I analyse the demonstrations of not-denumerability of Real Numbers to point out what are the fallaciousness' of these kind of proofs.

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1. INTRODUCTION

After his proof on denumerability of Rational Numbers, Cantor exposed his arguments about the not-denumerability of Real Numbers, by means of which he intended to recover the not denumerability of a real interval from his proof of denumerability of Rational Numbers. If a real interval must be considered continuous, Real Numbers must not to be denumerable since otherwise, reordering their sequence indexing so to follow the usual magnitude order[1], there would be "holes" left between any two terms[2] (in particular there would be problems with the relationship between Real Numbers and geometry). The proofs follow the same scheme of the demonstration of the denumerability of Rational Numbers. But, as I proved in my "Not Denumerability of Rational Numbers"[1], they are wrong since they use the term "all" and the shortcut "…" without any clear definition of their meaning, resulting only in a way to hide the correct deductions.

2. CANTOR'S FIRST UNCOUNTABILITY PROOF

This is one of the arguments, due to Cantor [2], to prove that Real Numbers are not countable.

> Cantor's first uncountability proof [3, p. 32]:
>
> Consider an infinite sequence of different real numbers $(a_\nu) = a_1, a_2, a_3, \ldots$ which is given by any rule, then we can find in any open interval (α, β) a number η (and, hence, infinitely many of such numbers) which is not a member of the sequence (a_ν).
>
> Take the first two members of sequence (a_ν) which fit into the given interval (α, β). They form the interval (α', β'). The first two members of sequence (a_ν) which fit into this interval (α', β') form the interval (α'', β'') and so on. The result is a sequence of nested intervals. Now there are only two possible cases.
>
> *Either* the number of intervals is finite. Inside the last one $(\alpha^{(\nu)}, \beta^{(\nu)})$ there cannot be more than one member of the sequence. Any other number of this interval $(\alpha^{(\nu)}, \beta^{(\nu)})$ can be taken as η.
>
> *Or* the number of intervals is infinite. Then both, the strictly increasing sequence $\alpha, \alpha', \alpha'', \ldots$ and the strictly decreasing sequence $\beta, \beta', \beta'', \ldots$ converge to different limits α^∞ and β^∞ or they converge to the same limit $\alpha^\infty = \beta^\infty$. $\alpha^\infty = \beta^\infty = \eta$ is not a member of sequence (a_ν). If $\alpha^\infty < \beta^\infty$, then any number of $(\alpha^\infty, \beta^\infty)$ satisfies the theorem. ▢

If the sequence is in magnitude order, then the proof is banal since between two real numbers there are infinite real numbers and even infinite rational numbers. So we can assume that Cantor meant an infinite not-ordered sequence of real numbers.

Let us analyse the steps of the proof:

The case "either" of the proof is clearly wrong since, the sequence being not-ordered, one cannot be sure that no more than one member in the sequence would fall in the interval without check them all, so the process cannot terminate after a finite number of steps[3].

In the case "or",

- if $\alpha^\infty < \beta^\infty$ then the statement is correct.
- If we have $\alpha^\infty = \beta^\infty$ then we must do some more considerations:

- First if we would consider the limit as potential, it would be never-ending and so undecidable, since we cannot check "all" the members of the sequence.

1. If one accepts the actual infinity as mathematically treatable, then one has "all" the numbers in the sequence and so the reordering must be accepted as possible. In the case one refuses that the actual infinity is mathematically treatable, as I do, then reordering an infinity of indexes is not possible since infinity is only in progress and does not even exist an end.

2. I want to stress here that I don't believe that a sum of zero-dimension entities like points can result in something that is different from zero. I stick with the statement from Aristoteles that considered a line made up by lines, planes by planes and volumes by volumes.

3. Also the remark of Prof. Mückenheim:

"We note that the "either"-case cannot occur if sequence (a_ν) contains at least all rational numbers because any interval $(\alpha^{(\nu)}, \beta^{(\nu)}) \subset \mathbb{Q}$ contains infinitely many and so at least two rational numbers forming the next interval $(\alpha^{(\nu+1)}, \beta^{(\nu+1)}) \subset \mathbb{Q}$."[3, p. 32]

- Otherwise we would reach the limits and they should belong to the sequence, since in $(\alpha^{(v)}, \beta^{(v)})$, $\alpha^{(v)}$ and $\beta^{(v)}$ are members of of (a_v) by construction; and in effect they do belong to the sequence since we have "all" the members of it, and therefore we can find its greatest member smaller than or equal to α^∞, and its smallest member greater than or equal to β^∞; that means that, if α^∞ and β^∞ would not be in the sequence, α^∞ would not be the limit of the increasing sequence and β^∞ would not be the limit of the deceasing sequence, contradicting the hypothesis. Since they belong to the sequence, the final step is the $(\alpha^\infty, \beta^\infty) \equiv (\alpha^\infty, \alpha^\infty) \equiv (\beta^\infty, \beta^\infty)$ interval, that is a null interval, with no number in it and this contradicts the statement of the theorem.

So we have that, depending on what the case is in the proof, we fall in different conclusions, i.e. the theorem is undecidable (with this kind of proof).

3. CANTOR'S SECOND UNCOUNTABILITY PROOF

Cantor's second uncountability proof also known as Cantor's second diagonal method [4], was presented using only two elements (or digits): *m*, *w*. Nowadays it is used to present it in an interval (0,1) with decimal numbers. This clearly does not invalidate the followings.

1. Theorem [5]: The set of all real numbers is uncountable.
   Proof: We restrict ourselves to (0, 1) and we assume we can "list" all the numbers of (0, 1).
   The first argument (presentation and historically is Cantor's):
   Our list:
   $x_1 = .a_{11}a_{12}a_{13}\cdots$
   $x_2 = .a_{21}a_{22}a_{23}\cdots$
   $\vdots$
   $x_k = .a_{k1}a_{k2}a_{k3}\cdots$
   $\vdots$

   Let $a^* = .a_1a_2a_3\cdots$ where $a_k = \begin{cases} 5 & a_{kk} \text{ even} \\ 2 & a_{kk} \text{ odd} \end{cases}$.
   Since $a^* \neq x_k$ for all k, a^* is not on the list, but $a^* \in (0, 1)$. □

Let us examine the proof.

Clearly the list is denumerably infinite by construction. And the assumption is that "all" the numbers are listed. But, if so, it is totally wrong to state that "$a^* \neq x_k$ for all k", since, whatever a_k be, there **can** be an infinite number of x_i with $a_{ik} = a_k$: between the number $x_k^{(1)}=.a_{k1}a_{k2}a_{k3}\cdots a_{k\,k-1}00\cdots$ and the number $x_k^{(2)}=.a_{k1}a_{k2}a_{k3}\cdots a_{k\,k-1}99\cdots$, there are "all" the numbers with different digits after the (k-1)th one, i.e.:

$.a_{i1}a_{i2}a_{i3}\cdots a_{i\,k-1}00\cdots$
$\vdots$
$.a_{i1}a_{i2}a_{i3}\cdots a_{i\,k-1}01\cdots$
$\vdots$
$.a_{i1}a_{i2}a_{i3}\cdots a_{i\,k-1}02\cdots$
$\vdots$
$.a_{i1}a_{i2}a_{i3}\cdots a_{i\,k-1}10\cdots$
$\vdots$
$.a_{i1}a_{i2}a_{i3}\cdots a_{i\,k-1}11\cdots$
$\vdots$
$.a_{i1}a_{i2}a_{i3}\cdots a_{i\,k-1}12\cdots$
$\vdots$
$\cdot a_{i1}a_{i2}a_{i3}\cdots a_{i\,k-1}90\cdots$
$\vdots$
$\cdot a_{i1}a_{i2}a_{i3}\cdots a_{i\,k-1}91\cdots$
$\vdots$
$.a_{i1}a_{i2}a_{i3}\cdots a_{i\,k-1}99\cdots$

So, whatever a_k is, there **can** be infinite x_i with the same digits $.a_1a_2a_3\cdots a_k$ than a^*. Modifying one digit in an infinite indeterminate sequence has no meaning.

The logic error in the demonstration is the implicit assumption that in the sequence x_k, $a_{ik} \neq a_{jk}$ $\forall i, j;\ i \neq j$[4].

4. See also: "Dark Numbers"[6].

4. ANOTHER ARGUMENT [5]

> Consider an infinite sequence of different real numbers $(a_\nu) = a_1, a_2, a_3, \ldots$ which is given by any rule, then we can find in the open interval (0, 1) a number η (and, hence, infinitely many of such numbers) which is not a member of the sequence (a_ν).
> Divide [0, 1] into [0, 1/3], [1/3, 2/3], and [2/3, 1]. One of these intervals does not contain x_1 (if x_1 is 1/3, pick [2/3, 1]).
> Call this interval $I_1 = [a_1, b_1]$. Divide I_1 into thirds and pick one that does not contain x_2. Call this interval $I_2 = [a_2, b_2]$. Repeating the process, we have $I_{k+1} \subset I_k$, and $l(I_k) = 1/3^k$. The set of left-hand endpoints, [a1, a2, …], is bounded above by 1. Thus by LUB has a supremum a*. But $a^* \leq b_k$ for all k. Thus $a^* \in [a_k, b_k]$ for all k and $a^* \neq x_k$ for all k. a* is not on our list. □

This is another case in which the indiscriminate and vague use of the term "all" leads to erroneous results. In fact, if the limit is treated as a limit, i.e. it is never reached, the proof never ends and so it is undecidable. Alternatively, if it is considered reachable, as usual when actual infinite is considered valid, then $\lim\limits_{k\to\infty} l(I_k)=0 \Rightarrow a_\infty \equiv b_\infty$ and $a^*=a_\infty \equiv b_\infty = x_{\infty-1} \equiv x_\infty$, since $x_{k-1} \overset{maybe}{\in} [a_k, b_k]$ and $\lim\limits_{k\to\infty} x_{k-1} \overset{maybe}{\in} [a_k, b_k] \Rightarrow x_\infty \overset{maybe}{\in} [a_\infty, b_\infty]=a_\infty \equiv b_\infty \Rightarrow x_\infty = a^*$ (note that $\lim\limits_{k\to\infty} x_{k-1} = \lim\limits_{k\to\infty} x_k = x_\infty$). So a* **may** be on the list and the theorem is still undecidable.

4.1. Remarks

If you find yourselves uncomfortable with the index (k-1) think that it is only a matter of convention. In the proof x_1 is connected to the interval [0, 1], which we could call I_0. But if we number the elements of the sequence starting from x_0, then we have that x_i will be connected with I_i. Alternatively we could call [0, 1] as I_1 and so on, and we would face with the same connection above.

5. CONCLUSIONS

Cantor's approach to infinity in mathematics flourished in the main stream of the new and strong interest in mathematics of the Nineteenth Century. The achievements of science and technology gave to researchers the confidence that human mind could explain all of the Nature laws. In mathematics there were many tentatives to formalise its foundations that lead to great enhancements. Among these tentatives there were some that failed but this is not any bad: there cannot be enhancements without mistakes. What is very important is to recognise them and to learn from them. But one great error is still in use and not corrected. The naif approach to actual infinity, introduced by Cantor with its demonstration of the Denumerability of Rational Numbers, reached by means of the use of not rigorous demonstrations and the acceptance of not consistency in the mathematical definitions [1, p. 8].

How it is seen in this paper, the not justified use of terms such as "all" or of shortcuts such as "…", can easily lead to inconsistent results.

6. REFERENCES

[1] Musmeci, R. *Not Denumerability of Rational Numbers*. (2021). <https://doi.org/10.5281/zenodo.4483594>

[2] Cantor, G. *Ueber Eine Eigenschaft Des Inbegriffs Aller Reellen Algebraischen Zahlen.* Journal für die reine und angewandte Mathematik **77**, 258-262 (1874). <https://eudml.org/doc/148238>

[3] Mückenheim, W. *Transfinity: A Source Book*. (2020). <https://www.hs-augsburg.de/~mueckenh/Transfinity/Transfinity/pdf>

[4] Cantor, G. *Über Eine Elementare Frage Der Mannigfaltigkeitslehre*. (1890).

[5] Burk, F. *Lebesgue Measure and Integration* (John Wiley & Sons, 2011).

[6] Mückenheim, W. *Dark Numbers*. <https://www.academia.edu/44503118/Dark_Numbers?source=swp_share>